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An Interpolation Formula for Harmonic Functions on the Set of Integers

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If u is an entire harmonic function of exponential type less than π satisfying $u(n, 0) = u_y(n, 0) = 0$ for all integers n, then it follows from a result of Zeilberger that u is identically zero. The main result in this article is to give a series representation of any entire harmonic function u of exponential type $\tau \leqslant \pi$ with $u(x, 0)$ and $u_y(x, 0)$ in $L^p(-\infty, \infty)$, for some $p > 0$, in terms of the values $u(n, 0)$ and $u_y(n, 0)$, $n = 0, 1, \ldots$. A method of construction of a basis is given. This technique can be extended to other related problems.

1. INTRODUCTION AND RESULTS

An entire harmonic function $u(x, y)$ is a real-valued function, harmonic in the plane $\mathbb{R}^2$. It is an entire harmonic function of exponential type τ, if $|u(x, y)| \leqslant e^{(\tau+\epsilon)|(x,y)|}$ for all large values of $|(x, y)| = \sqrt{x^2 + y^2}$ where $\epsilon > 0$ is arbitrary. By Carathéodory's inequality [4; *p.* 3], $u(x, y)$ is entire harmonic of exponential type τ if and only if $u(x, y) = \operatorname{Re} f(z)$, $z = x + iy$, where f is an entire (analytic) function of exponential type τ. In [3], Boas pointed out that at least we need to know the function values on a two-dimensional set in $\mathbb{R}^2$ in order to uniquely determine an entire harmonic function. For instance, the function $u(x, y) = y$ is zero for all $y = 0$. He also proved in [3] that if $u(x, y)$ is entire harmonic and of exponential type type $\tau < \pi$, then $u(x, y) \equiv 0$ if it is zero at the lattice points $(n, 0)$ and $(n, 1)$, $n = 0, \pm 1, \pm 2, \ldots$. Several interesting problems were posed in [3]. One of these problems was to construct $u(x, y)$ from its values at these lattice points. This question was answered by Ching and Chui in [5], in which $u(x, 0)$

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and $u(x, 1)$ are both assumed to belong to $L^2(-\infty, \infty)$. Later, Anderson proved the convergence of the series representation introduced in [5] under weaker hypotheses on $u(x, 0)$ and $u(x, 1)$. Another problem in [3] was to generalize the uniqueness results in [3] to higher dimensions. This was done in [2, 7]. As a consequence, it follows that the values $u(n, 0)$ and $u_y(n, 0)$, $n = 0, 1, \ldots$, on a one-dimensional set are sufficient to determine u provided that the type of τ of u is less than π. That is, the following result is obtained.

PROPOSITION 1. *Let $u(x, y)$ be an entire harmonic function of exponential type $\tau < \pi$ such that $u(n, 0) = u_y(n, 0) = 0$, $n = 0, \pm 1, \pm 2, \ldots$ Then $u(x, y) \equiv 0$.*

This result is sharp in the sense that $\tau < \pi$ cannot be replaced by $\tau \leqslant \pi$, because of the example $(c_1 e^{\pi y} + c_2 e^{-\pi y}) \sin \pi x$. We also remark that the function $u(x, y) = y$ satisfies $u(n, 0) = u_x(n, 0) = 0$ for $n = 0, \pm 1, \ldots$, so that the hypotheses on the normal derivative u_y cannot be replaced by that on the tangential derivative. We will give a very simple function-theoretic proof of this uniqueness result. This proof will also facilitate our derivation of our series representation theorem.

Note that the functions

$$\alpha_n(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh ty\, e^{it(x-n)}\, dt$$

and

$$\beta_n(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh ty}{t}\, e^{it(x-n)}\, dt$$

are entire harmonic functions of exponential type π and satisfy

$$\alpha_n(m, 0) = \delta_{m,n}, \qquad \frac{\partial}{\partial y} \alpha_n(m, 0) = 0,$$

and

$$\beta_n(m, 0) = 0, \qquad \frac{\partial}{\partial y} \beta_n(m, 0) = \delta_{m,n}, \tag{1}$$

for all integers m and n. We use these functions as a basis to solve the representation problem. Namely, we have the following result.

THEOREM 1. *Let $u(x, y)$ be an entire harmonic function of exponential type $\leqslant \pi$ such that $u(x, 0)$ and $u_y(x, 0)$ belong to $L^p(-\infty, \infty)$, for some p, $0 < p < \infty$. Then*

$$u(x, y) = \sum_{n=-\infty}^{\infty} u(n, 0)\, \alpha_n(x, y) + \sum_{n=-\infty}^{\infty} u_y(n, 0)\, \beta_n(x, y), \tag{2}$$

where the series converges uniformly on every compact subset of $\mathbb{R}^2$.

We emphasize that while Proposition 1 holds only for entire harmonic functions of exponential type less than π, Theorem 1 holds for those of type $\leqslant \pi$. Of course, for $y = 0$ the function $(c_1 e^{\pi y} + c_2 e^{-\pi y}) \sin \pi x$ is not in L^p, $0 < p < \infty$. This example also shows that $0 < p < \infty$ cannot be replaced by $p = \infty$ in Theorem 1. For $p = 2$, following Ching and Chui [5], it can be proved that the convergence of the series in (2) is uniform on every strip $|y| \leqslant K < \infty$. We will give a method of construction of the basis functions α_n and β_n. This method can be extended to other related problems.

2. Proof of the Results

Let $u(x, y)$ be an entire harmonic function of exponential type $\tau < \pi$ such that $u(n, 0) = u_y(n, 0) = 0, n = 0, \pm 1, \ldots$. We have to prove that $u(x, y) \equiv 0$. Let $v(x, y)$ be a harmonic conjugate of $u(x, y)$ and $f(z) = u(z) + iv(z)$. Also, set $F(z) = f(z) + \overline{f(\bar{z})}$. Then $F(z)$ is an entire function of exponential type $\tau < \pi$ and reduces to $2u(x, 0)$ on the real axis. Thus, $F(z)$ vanishes at all the integers. By Carlson's theorem, $F(z)$ vanishes identically. Now, consider the entire function $G(z) = f'(z) - \overline{f'(\bar{z})}$. $G(z)$ is of exponential type $\tau < \pi$ and reduces to $2iv_x(x, 0)$ on the real axis. But $v_x(x, y) = -u_y(x, y)$ so that $G(x) = -2iu_y(x, 0)$. But $u_y(n, 0) = 0$, $n = 0, \pm 1, \ldots$, yields $G(n) = 0$, $n = 0, \pm 1, \ldots$. Again by Carlson's theorem, $G(z)$ is identically zero.

From the identities $F(z) = G(z) = 0$ and using the Cauchy–Riemann equations, we have $u_x(x, -y) = -u_x(x, y)$ and $u_x(x, -y) = u_x(x, y)$, so that $u_x(x, y)$ is constant for fixed y. Similarly, we also have $u_y(x, -y) = u_y(x, y)$ and $u_y(x, -y) = -u_y(x, y)$ so that $u_y(x, y)$ is also constant when y is fixed. Hence by a standard argument, we have $u(x, y) = c_1 y + c_2$ for some constants c_1 and c_2. But the hypothesis $u(0, 0) = u(1, 0) = 0$ implies that $c_1 = c_2 = 0$, or $u(x, y) \equiv 0$. This completes the proof of Proposition 1. To prove the representation theorem, we first give a method of construction of the basis functions α_n and β_n.

In view of the basis functions obtained in [5], it is natural to consider the basis functions α_n and β_n of the form

$$h_n(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(t, y)\, e^{it(x-n)}\, dt.$$

If h_n is entire harmonic, then the kernel $k(t, y)$ must satisfy the equation $k_{yy} - t^2 k = 0$ for all t and y. Fix t and consider $K(\cdot) = k(t, \cdot)$ as a function of the second variable. By choosing $k(t, 0) \equiv 1$, then $h_n(m, 0) = \delta_{m,n}$ already. Hence, if we require $\partial h_n(m, 0)/\partial y = 0$ for all m, we have

$$\int_{\pi}^{\pi} k_y(t, 0)\, e^{itj}\, dt = 0$$

for all integers j, so that $k_y(t, 0) = 0$. That is, we have the initial valued problem

$$K'' - t^2 K = 0,$$
$$K(0) = 1, \qquad K'(0) = 0.$$

The solution is $k(t, y) = K(y) = \cosh ty$. That is, the basis function $\alpha_n(x, y)$ satisfying $\alpha_n(m, 0) = \delta_{n,m}$ and $\partial\alpha_n(m, 0)/\partial y = 0$ for all m is

$$\alpha_n(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh ty e^{it(x-n)}\, dt.$$

Similarly, the basis function $\beta_n(x, y)$ satisfying $\beta_n(m, 0) = 0$, $\partial\beta_n(m, 0)/\partial y = \delta_{m,n}$ for all m is obtained to be

$$\beta_n(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sinh ty}{t} e^{it(x-n)}\, dt.$$

To complete the proof of Theorem 1, we need several lemmas.

LEMMA 1. *For all* $(x, y) \in \mathbb{R}^2$ *and* $n \neq 0$,

$$| \alpha_n(x, y)| \leqslant \left(\frac{1}{\pi} + | x | + | y |\right) e^{\pi|y|} \, | n |^{-1} \tag{3}$$

$$| \beta_n(x, y)| \leqslant \left(1 + \frac{1}{\pi} + \pi | y | + y^2 + | xy |\right) e^{\pi|y|} \, | n |^{-1}, \tag{4}$$

and $| \alpha_0(x, y)| \leqslant e^{\pi|y|}$, $| \beta_0(x, y)| \leqslant | y | e^{\pi|y|}$.

Let $z = x + iy$ be fixed and consider the functions $j(t) = e^{itx} \cosh ty$ and $k(t) = e^{itx} \sinh ty/t$. Then $\alpha_n(x, y)$ and $\beta_n(x, y)$ are the nth Fourier coefficients of $j(t)$ and $k(t)$, respectively. We first prove that for all $| t | \leqslant \pi$, $j(t)$ and $k(t)$ satisfy the following inequalities:

$$| j(t)| \leqslant e^{\pi|y|}, \tag{5}$$

$$| j'(t)| \leqslant (| x | + | y |)\, e^{\pi|y|}, \tag{6}$$

$$| k(t)| \leqslant | y | \, e^{\pi|y|}, \tag{7}$$

and

$$| k'(t)| \leqslant (1 + \pi | y | + y^2 + | xy |)\, e^{\pi|y|}. \tag{8}$$

Inequalities (5) and (6) are immediate from the definitions of $j(t)$. For $| t | \leqslant \pi$, we have

$$\left| \frac{\sinh ty}{t} \right| = \left| \sum_{n=0}^{\infty} \frac{(ty)^{2n}}{(2n+1)!} \right| | y |$$
$$\leqslant | y | \cosh | ty | \leqslant | y | \, e^{\pi|y|}, \tag{9}$$

which gives (7). To prove (8), we set $l(t) = \sinh ty/t$ and note that $l'(t) = (ty \cosh ty - \sinh ty)/t^2$. Hence, for $1 \leqslant |t| \leqslant \pi$, $|l'(t)| \leqslant (\pi |y| + 1) e^{\pi|y|}$, and for $|t| \leqslant 1$, we have

$$\begin{aligned} |l'(t)| &= \frac{1}{t^2} \left| \sum_{n=0}^{\infty} \frac{(ty)^{2n+1}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(ty)^{2n+1}}{(2n+1)!} \right| \\ &\leqslant \frac{1}{t^2} \left| \sum_{n=1}^{\infty} \frac{2n}{(2n+1)!} (ty)^{2n+1} \right| \\ &\leqslant |y|^2 \,|\sinh |ty|\,| \leqslant y^2 e^{|y|} \leqslant y^2 e^{\pi|y|}. \end{aligned}$$

These estimates give

$$|l'(t)| \leqslant (1 + \pi |y| + y^2)\, e^{\pi|y|}. \tag{10}$$

Using (9) and (10), we therefore have

$$\begin{aligned} |k'(t)| &\leqslant |x|\, |l(t)| + |l'(t)| \\ &\leqslant (|xy| + 1 + \pi |y| + y^2)\, e^{\pi|y|}, \end{aligned}$$

which is (8). We now return to derive the estimates for $\alpha_n(x, y)$ and $\beta_n(x, y)$. The estimates for $n = 0$ are trivial. For $n \neq 0$, we have

$$\begin{aligned} |\alpha_n(x, y)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} j(t)\, e^{-int}\, dt \right| \\ &\leqslant |\sin \pi(n - x) \cosh \pi y/\pi n| \\ &\quad + \left| \frac{1}{2\pi n} \int_{-\pi}^{\pi} j'(t)\, e^{-int}\, dt \right| \\ &\leqslant \left(\frac{1}{\pi} + |x| + |y| \right) e^{\pi|y|}/|n| \end{aligned}$$

by using (6). This gives (3). To estimate $\beta_n(x, y)$, $n \neq 0$, we have

$$\begin{aligned} |\beta_n(x, y)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} k(t)\, e^{-int}\, dt \right| \\ &\leqslant |\sinh (\pi y) \sin \pi(n - x)/\pi n| \\ &\quad + \left| \frac{1}{2\pi_n} \int_{-\pi}^{\pi} k'(t)\, e^{-int}\, dt \right| \\ &\leqslant \left(\frac{1}{\pi} + 1 + \pi |y| + y^2 + |xy| \right) e^{\pi|y|}/|n| \end{aligned}$$

by using (8). This completes the proof of Lemma 1.

We also need the following lemma which was proved in [1].

LEMMA 2. *Let $u(x, y)$ be an entire harmonic function of exponential type $\tau \leqslant \pi$ such that $u(x, 0)$ and $u_y(x, 0)$ are both in $L^p(-\infty, \infty)$, $0 < p < \infty$. Then $u(x, n)$ satisfies the following*:

$$\sum_{n \neq 0} \left| \frac{u(n, 0)}{n} \right| < \infty, \tag{11}$$

$$\sum_{n \neq 0} \left| \frac{u_y(n, 0)}{n} \right| < \infty, \tag{12}$$

$$u(x, 0) = o(1) \quad \textit{and} \quad u_y(x, 0) = o(1) \quad \textit{as} \quad |x| \to \infty. \tag{13}$$

We are now ready to prove Theorem 1. Let $u(x, y)$ satisfy the hypotheses in the theorem. For any compact set K in $\mathbb{R}^2$, Lemma 1 yields

$$\begin{aligned} |u(n, 0)\, \alpha_n(x, y)| &\leqslant \left(\frac{1}{\pi} + |x| + |y|\right) e^{\pi|y|} \left| \frac{u(n, 0)}{n} \right| \\ &\leqslant C_K \left| \frac{u(n, 0)}{n} \right| \end{aligned}$$

for all $(x, y) \in K$, where $C_K < \infty$ is some constant depending only on K. A similar estimate holds for $u_y(n, 0)\, \beta_n(x, y)$. Hence, by Lemma 2, the series

$$\sum_{n=-\infty}^{\infty} u(n, 0)\, \alpha_n(x, y) + u_y(n, 0)\, \beta_n(x, y)$$

converges uniformly on every compact subset of $\mathbb{R}^2$ to some entire harmonic function $w(x, y)$. From the same estimates above, it is clear that $w(x, y)$ is of exponential type $\tau \leqslant \pi$. Let $U(x, y) = u(x, y) - w(x, y)$. It is sufficient to prove that $U(x, y) \equiv 0$. Using the proof given in [1], that is, by a consecutive application of the Plancherel–Pólya and the Riemann–Stieltjes theorems, we conclude that $U(x, 0) = U_y(x, y) = 0$ for all x. As in the proof of the proposition, let $h(z)$ be an entire function with $\operatorname{Re} h(z) = U(x, y)$, $z = x + iy$, and set $H(z) = h(z) + \overline{h(\bar{z})}$. Then $H(x) = 2U(x, 0) = 0$ for all real x, so that $H(z) \equiv 0$. Similarly, let $L(z) = h'(z) - \overline{h'(\bar{z})}$. Then $L(x) = 2iU_y(x, 0) = 0$ by using the Cauchy–Riemann equations. Thus, $L(z) \equiv 0$. Write $h(z) = \sum a_n z^n$. Since $H(z) = h(z) + \overline{h(\bar{z})} \equiv 0$, we have $a_n + \bar{a}_n = 0$ for all n. Also, $L(z) = h'(z) - \overline{h'(\bar{z})} \equiv 0$ implies $na_n - n\bar{a}_n = 0$ for all n. We conclude, therefore, that a_0 is pure imaginary and $a_n = 0$ for $n = 1, 2, \ldots$. Hence, $U(x, y) = \operatorname{Re} h(z) \equiv 0$, completing the proof of the theorem.

3. Final Remarks

By using standard techniques in classical function theory, the uniqueness result in Proposition 1 can be improved. For example, the following uniqueness result is obtained in [6].

PROPOSITION 2. *If u is an entire harmonic function of exponential type* π *such that* $|u(x, y)|$, $|u_x(x, y)|$, $|u_y(x, y)| \leqslant Ae^{\pi|(x,y)|}$, *and* $u(n, 0) = u_y(n, 0) = 0$ *for* $n = 0, \pm 1, \ldots$, *then* $u(x, y) = (c_1 e^{\pi y} + c_2 e^{-\pi y}) \sin \pi x$.

Similarly, the techniques used in this paper can be extended to give other related uniqueness and representation results (cf. [6]).

References

1. K. F. Anderson, On the representation of harmonic functions by their values on lattice points, *J. Math. Anal. Appl.* **49** (1975), 692–695.
2. D. H. Armitage, Uniqueness theorems for harmonic functions which vanish at lattice points, *J. Approximation Theory* **26** (1979), 259–268.
3. R. P. Boas, Jr., A uniqueness theorem for harmonic functions, *J. Approximation Theory* **5** (1972), 425–427.
4. R. P. Boas, Jr., "Entire Functions," Academic Press, New York, 1954.
5. C. H. Ching and C. K. Chui, A representation formula for harmonic functions. *Proc. Amer. Math. Soc.* **39** (1973), 349–352.
6. G. A. Roberts, "Uniqueness and Interpolation of Entire Harmonic Functions," Ph.D. dissertation, Texas A&M Univ., College Station, 1979.
7. D. Zeilberger, Uniqueness theorems for harmonic functions of exponential growth, *Proc. Amer. Math. Soc.* **61** (1976), 335–340.